10 General Relativistic Models for Space-time Coordinates and Equations of Motion

10.1 Time Coordinates

IAU resolution A4 (1991) set the framework presently used to define the barycentric reference system (BRS) and the geocentric reference system (GRS). Its third recommendation defined Barycentric Coordinate Time (TCB) and Geocentric Coordinate Time (TCG) as time coordinates of the BRS and GRS, respectively. In the fourth recommendation another time coordinate is defined for the GRS, Terrestrial Time (TT). This framework was further refined by the IAU Resolutions B1.3 and B1.4 (2000) to provide consistent definitions for the coordinates and metric tensor of the reference systems at the full post-Newtonian level (Soffel, 2000). At the same time IAU Resolution B1.5 (2000) applied this framework to time coordinates and time transformations between reference systems, and IAU Resolution B1.9 (2000) re-defined Terrestrial Time (Petit, 2000). TT differs from TCG by a constant rate, $dTT/dTCG = 1 - L_G$, where L_G is a defining constant. The value of L_G (see Table 1.1) has been chosen to provide continuity with the former definition of TT, *i.e.* that the unit of measurement of TT agrees with the SI second on the geoid. The difference between TCG and TT may be expressed as

$$\text{TCG} - \text{TT} = \text{L}_\text{G} \times (\text{MJD} - 43144.0) \times 86400 \text{ s},$$

where MJD refers to the modified Julian date of International Atomic Time (TAI). TAI is a realization of TT, apart from a constant offset:

TT = TAI + 32.184 s.

Before 1991, previous IAU definitions of the time coordinates in the barycentric and geocentric frames required that only periodic differences exist between Barycentric Dynamical Time (TDB) and Terrestrial Dynamical Time (TDT) (Kaplan, 1981). As a consequence, the spatial coordinates in the barycentric frame had to be rescaled to keep the speed of light unchanged between the barycentric and the geocentric frames (Misner, 1982; Hellings, 1986). Thus, when barycentric (or TDB) units of length were compared to geocentric (or TDT) units of length, a scale difference, L, appeared (see also Chapter 1). This is no longer required with the use of the TCG/TCB time scales.

The relation between TCB and TDB is linear. It may be given in seconds by

$$\text{TCB} - \text{TDB} = \text{L}_\text{B} \times (\text{MJD} - 43144.0) \times 86400 + P_0, \quad P_0 \approx 6.55 \times 10^{-5} s.$$

However, since no precise definition of TDB exists, there is no definitive value of L_B and such an expression should be used with caution.

Figure 10.1 shows graphically the relationships between the time scales. See *IERS Technical Note 13*, pages 137–142 for copies of the IAU Resolution A4 (1991) and Appendix 1 of this volume for copies of the resolutions of the 24th IAU General Assembly (2000) relating to reference systems and time coordinates.

The difference between Barycentric Coordinate Time (TCB) and Geocentric Coordinate Time (TCG) involves a four-dimensional transformation,

$$\text{TCB} - \text{TCG} = c^{-2} \left\{ \int_{t_0}^{t} [\frac{v_e^2}{2} + U_{ext}(\vec{x}_e)] dt + \vec{v}_e \cdot (\vec{x} - \vec{x}_e) \right\} + O(c^{-4}),$$

where $\vec{x}_e$ and $\vec{v}_e$ denote the barycentric position and velocity of the Earth's center of mass, $\vec{x}$ is the barycentric position of the observer and

U_{ext} is the Newtonian potential of all of the solar system bodies apart from the Earth evaluated at the geocenter. In this formula, t is TCB and t_0 is chosen to be consistent with 1977 January 1, $0^{\mathrm{h}}0^{\mathrm{m}}0^{\mathrm{s}}$ TAI. This formula is only valid to within the neglected terms, of order 10^{-16} in rate, and IAU Resolution B1.5 (2000) provides formulas to compute the $O(c^{-4})$ terms within given uncertainty limits.
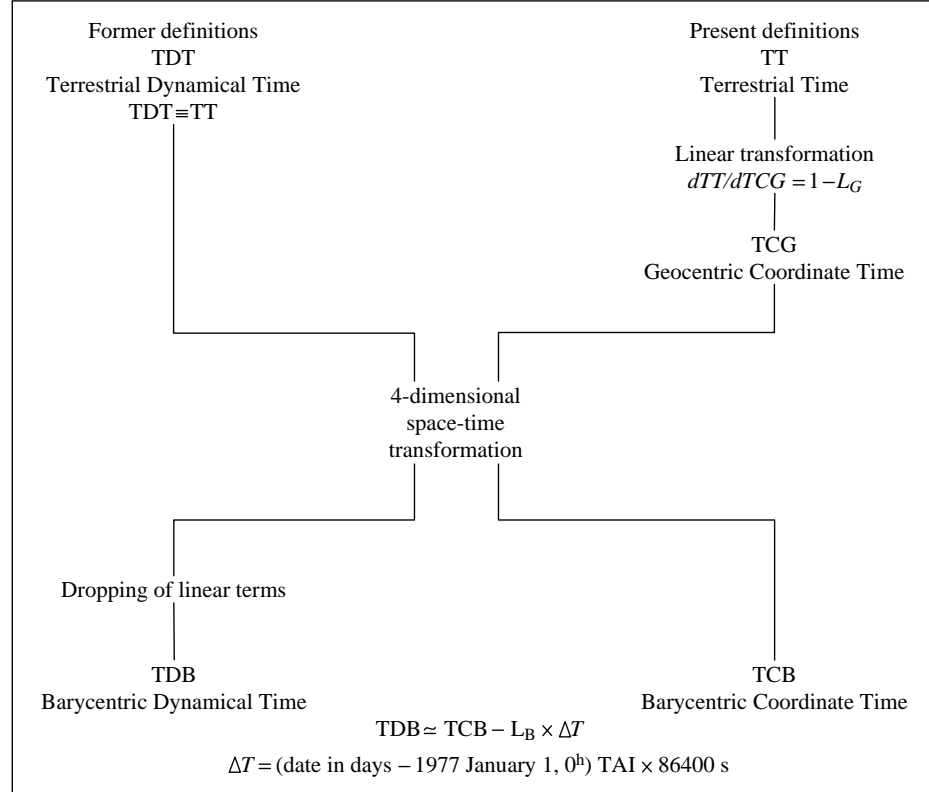

Fig. 10.1   Relations between time scales.

An approximation of the TCB−TCG formula is given by

$$(\mathrm{TCB} - \mathrm{TCG}) = \frac{L_C \times (TT - TT_0) + P(TT) - P(TT_0)}{(1 - L_B)} + c^{-2}\vec{v_e}\cdot(\vec{x}-\vec{x_e})$$

where TT_0 corresponds to JD 2443144.5 TAI (1977 January 1, 0 $^{\mathrm{h}}$) and where the values of L_C and L_B may be found in Table 1.1. Periodic terms denoted by $P(TT)$ have a maximum amplitude of around 1.6 ms and can be evaluated by the "FB" analytical model (Fairhead and Bretagnon, 1990; Bretagnon 2001). Alternately, $P(TT) - P(TT_0)$ may be provided by a numerical time ephemeris such as TE405 (Irwin and Fukushima, 1999), which provides values with an accuracy of 0.1 ns from 1600 to 2200. Irwin (2003) has shown that TE405 and the 2001 version of the FB model differ by less than 15 ns over the years 1600 to 2200 and by only a few ns over several decades around the present time. Finally a series, HF2002, providing the value of $L_C \times (TT - TT_0) + P(TT) - P(TT_0)$ as a function of TT over the years 1600–2200 has been fit (Harada and Fukushima, 2002) to TE405. This fit differs from TE405 by less than 3 ns over the years 1600–2200 with an RMS error of 0.5 ns. Note that in this section on the computation of TCB−TCG, TT is used as a time argument while the actual argument of the different realizations is T_{eph} (see Chapter 3). The resulting error in TCB−TCG is at most approximately 20 ps.

The time ephemeris TE405 is available in a Chebyshev form at $<^{19}>$. The 2001 version of the FB model is available at $<^{20}>$, where the files of interest are fb2001.f, fb2001.dat, fb2001.in, fb2001.out, and README.fb2001.f. The HF2002 model is available in the same directory, where the files of interest are Xhf2002.f, HF2002.DAT and hf2002.out.

10.2   Equations of Motion for an Artificial Earth Satellite [21]

The relativistic treatment of the near-Earth satellite orbit determination problem includes corrections to the equations of motion, the time transformations, and the measurement model. The two coordinate systems generally used when including relativity in near-Earth orbit determination solutions are the solar system barycentric frame of reference and the geocentric or Earth-centered frame of reference.

Ashby and Bertotti (1986) constructed a locally inertial E-frame in the neighborhood of the gravitating Earth and demonstrated that the gravitational effects of the Sun, Moon, and other planets are basically reduced to their tidal forces, with very small relativistic corrections. Thus the main relativistic effects on a near-Earth satellite are those described by the Schwarzschild field of the Earth itself. This result makes the geocentric frame more suitable for describing the motion of a near-Earth satellite (Ries *et al.*, 1989). Later on, two advanced relativistic formalisms have been elaborated to treat the problem of astronomical reference systems in the first post-Newtonian approximation of general relativity. One formalism is due to Brumberg and Kopeikin (Kopeikin, 1988; Brumberg and Kopeikin, 1989; Brumberg, 1991) and another one is due to Damour, Soffel and Xu (Damour, Soffel, Xu, 1991, 1992, 1993, 1994). These allow a full post-Newtonian treatment (Soffel, 2000) and form the basis of IAU Resolutions B1.3 and B1.4 (2000).

The relativistic correction to the acceleration of an artificial Earth satellite is

$$
\begin{aligned}
\Delta \ddot{\vec{r}} = \ & \frac{GM_E}{c^2 r^3} \left\{ [2(\beta + \gamma)\frac{GM_E}{r} - \gamma \dot{\vec{r}} \cdot \dot{\vec{r}}] \vec{r} + 2(1+\gamma)(\vec{r} \cdot \dot{\vec{r}})\dot{\vec{r}} \right\} + \\
& (1+\gamma)\frac{GM_E}{c^2 r^3} \left[\frac{3}{r^2}(\vec{r} \times \dot{\vec{r}})(\vec{r} \cdot \vec{J}) + (\dot{\vec{r}} \times \vec{J}) \right] + \\
& \left\{ (1+2\gamma) \left[\dot{\vec{R}} \times \left(\frac{-GM_S \vec{R}}{c^2 R^3} \right) \times \dot{\vec{r}} \right] \right\},
\end{aligned}
\tag{1}
$$

where

c = speed of light,

β, γ = PPN parameters equal to 1 in General Relativity,

$\vec{r}$ is the position of the satellite with respect to the Earth,

$\vec{R}$ is the position of the Earth with respect to the Sun,

$\vec{J}$ is the Earth's angular momentum per unit mass
($|\vec{J}| \cong 9.8 \times 10^8 \mathrm{m^2/s}$), and

GM_E and GM_S are the gravitational coefficients of the Earth and Sun, respectively.

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[19] ftp://astroftp.phys.uvic.ca in the directory /pub/irwin/tephemeris
[20] ftp://maia.usno.navy.mil in the directory /conv2000/chapter10/software
[21] The IAU Resolutions B1.3 and B1.4 (2000) and references therein now provide a consistent framework for the definition of the geocentric and barycentric reference systems at the full post-Newtonian level using harmonic coordinates. The equations of motion for spherically-symmetric and uniformly rotating bodies in these systems are the same as those previously derived in a Parametrized Post-Newtonian system.

The effects of Lense-Thirring precession (frame-dragging), geodesic (de Sitter) precession have been included. The relativistic effects of the Earth's oblateness have been neglected here but, if necessary, they could be included using the full post-Newtonian framework of IAU Resolutions B1.3 and B1.4 (2000). The independent variable of the satellite equations of motion may be, depending on the time transformation being used, either TT or TCG. Although the distinction is not essential to compute this relativistic correction, it is important to account for it properly in the Newtonian part of the acceleration.

10.3   Equations of Motion in the Barycentric Frame

(see footnote 21 preceeding page)

The n-body equations of motion for the solar system frame of reference (the isotropic Parameterized Post-Newtonian system with Barycentric Coordinate Time (TCB) as the time coordinate) are required to describe the dynamics of the solar system and artificial probes moving about the solar system (for example, see Moyer, 1971). These are the equations applied to the Moon's motion for Lunar Laser Ranging (Newhall *et al.*, 1987). In addition, relativistic corrections to the laser range measurement, the data timing, and the station coordinates are required (see Chapter 11).

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