

# Orbit Perturbations due to Relativistic Corrections

Urs Hugentobler  
August 18, 2008

The relativistic correction to the acceleration of an artificial Earth satellite is [IERS Conventions 2003]

$$\begin{aligned} \Delta \ddot{\vec{r}} = & \frac{GM_E}{c^2 r^3} \left\{ [2(\beta + \gamma) \frac{GM_E}{r} - \gamma \vec{r} \cdot \ddot{\vec{r}}] \vec{r} + 2(1 + \gamma) (\vec{r} \cdot \ddot{\vec{r}}) \vec{r} \right\} + \\ & (1 + \gamma) \frac{GM_E}{c^2 r^3} \left[ \frac{3}{r^2} (\vec{r} \times \ddot{\vec{r}}) (\vec{r} \cdot \vec{J}) + (\ddot{\vec{r}} \times \vec{J}) \right] + \\ & \left\{ (1 + 2\gamma) \left[ \ddot{\vec{R}} \times \left( \frac{-GM_S \vec{R}}{c^2 R^3} \right) \right] \times \vec{r} \right\} \end{aligned} \quad (1)$$

Gaussian perturbation equations [e.g. Beutler, 2005]:

$$\dot{a} = \sqrt{\frac{p}{GM}} \frac{2a}{1 - e^2} \left[ e \sin \nu \cdot R + \frac{p}{r} \cdot S \right] \quad (2)$$

$$\dot{e} = \sqrt{\frac{p}{GM}} [\sin \nu \cdot R + (\cos \nu + \cos E) \cdot S] \quad (3)$$

$$\dot{i} = \frac{r \cos u}{na^2 \sqrt{1 - e^2}} \cdot W \quad (4)$$

$$\dot{\Omega} = \frac{r \sin u}{na^2 \sqrt{1 - e^2} \sin i} \cdot W \quad (5)$$

$$\dot{\omega} = \frac{1}{e} \sqrt{\frac{p}{GM}} \left[ -\cos \nu \cdot R + \left( 1 + \frac{r}{p} \right) \sin \nu \cdot S \right] - \dot{\Omega} \cos i \quad (6)$$

$$\dot{M}_0 = \frac{1 - e^2}{nae} \left[ \left( \cos \nu - 2e \frac{r}{p} \right) \cdot R - \left( 1 + \frac{r}{p} \right) \sin \nu \cdot S \right] - \frac{3n}{2a} (t - t_0) \dot{a} \quad (7)$$

In the following we will consider circular orbits ( $e = 0$ ). The three perturbing accelerations  $R$ ,  $S$  and  $W$  in radial, alongtrack and crosstrack directions are then pointing parallel to the vectors  $\vec{r}$ ,  $\dot{\vec{r}}$  and  $\vec{r} \times \dot{\vec{r}}$  respectively. We will only be interested in secular perturbations.

With proper selection of the coordinate frame we may represent position, velocity, and angular momentum vectors for a circular orbit in the following way:

$$\begin{aligned} \vec{r} &= a(\cos \nu, \sin \nu, 0) \\ \vec{v} &= an(-\sin \nu, \cos \nu, 0) \\ \vec{r} \times \vec{v} &= a^2 n(0, 0, 1) \end{aligned} \quad (8)$$

In addition we may use  $u = \nu$ . The unit vectors in radial, alongtrack, and crosstrack directions can be written as

$$\begin{aligned} \vec{e}_R &= (\cos \nu, \sin \nu, 0) \\ \vec{e}_S &= (-\sin \nu, \cos \nu, 0) \\ \vec{e}_W &= (0, 0, 1) \end{aligned} \quad (9)$$

## 1 Schwarzschild

The Schwarzschild acceleration is for a circular orbit using  $GM = n^2 a^3$  reads (see (1))

$$\Delta \ddot{\vec{r}}_{\text{Sch}} = \frac{GM}{c^2 a^3} \left[ 4 \frac{GM}{a} - v^2 \right] \vec{r} = \frac{GM}{c^2 a^3} \left[ 4 \frac{GM}{a} - \frac{GM}{a} \right] \vec{r} = 3 \frac{(GM)^2}{c^2 a^4} \cdot \vec{r}, \quad (10)$$

The accelerations in radial, alongtrack, and crosstrack directions are thus

$$R = \vec{e}_R \cdot \Delta \ddot{\vec{r}}_{\text{Sch}} = 3 \frac{(GM)^2}{c^2 a^3}, \quad S = \vec{e}_S \cdot \Delta \ddot{\vec{r}} = 0, \quad W = \vec{e}_W \cdot \Delta \ddot{\vec{r}} = 0. \quad (11)$$

A circular orbit thus feels only a constant outward radial acceleration. This is equivalent to a change in  $GM$ :

$$\vec{e}_R \cdot \ddot{\vec{r}}_{\text{tot}} = -\frac{GM}{a^2} + R = -\frac{GM}{a^2} \left(1 - \frac{Ra^2}{GM}\right) = -\frac{GM'}{a^2}. \quad (12)$$

Keeping the revolution period unchanged and using Kepler's third law

$$a'^3 n^2 = GM' \quad \text{with} \quad a' = a + \Delta a \quad (13)$$

we get to first order

$$\Delta a = -\frac{1}{3} \frac{Ra^3}{GM}. \quad (14)$$

Using  $R$  from (11) we obtain a decrease of the semimajor axis of

$$\Delta a = -\frac{GM}{c^2} = -4.4\text{mm} \quad (15)$$

that is independent of  $a$ !

Since the Schwarzschild acceleration is in-plane it causes no perturbations in the orientation of the orbital plane.

## 2 Lense-Thirring

The Lense-Thirring effect describes the frame dragging in the vicinity of the Earth due to the rotating Earth. The corresponding frame precession is proportional to the Earth's angular momentum  $J$  and decreases with the distance from the Earth. The corresponding acceleration reads

$$\Delta \ddot{\vec{r}}_{\text{LT}} = 2 \frac{GM}{c^2 r^3} \left[ \frac{3}{r^2} (\vec{r} \times \vec{v})(\vec{r} \cdot \vec{J}) + (\vec{v} \times \vec{J}) \right]. \quad (16)$$

Obviously it includes an out-of-plane but no alongtrack acceleration. In the reference frame defined by (8) the Earth's angular momentum vector may be written as

$$\vec{J} = J(0, \sin i, \cos i). \quad (17)$$

For a circular orbit we may thus write

$$\vec{r} \cdot \vec{J} = aJ \sin i \sin \nu, \quad \vec{v} \times \vec{J} = anJ(\cos \nu \cos i, \sin \nu \cos i, -\sin \nu \sin i). \quad (18)$$

We may thus write the decomposition of the acceleration into radial, alongtrack, and crosstrack components as

$$\begin{aligned} R &= \Delta \ddot{\vec{r}}_{\text{LT}} \cdot \vec{e}_R = 2 \frac{GM}{c^2 a^4} \vec{r} \cdot (\vec{v} \times \vec{J}) = 2 \frac{GM}{c^2 a^4} \vec{J} \cdot (\vec{r} \times \vec{v}) = 2 \frac{GM}{c^2 a^2} nJ \cos i \\ S &= \Delta \ddot{\vec{r}}_{\text{LT}} \cdot \vec{e}_S = 0 \\ W &= \Delta \ddot{\vec{r}}_{\text{LT}} \cdot \vec{e}_W = 2 \frac{GM}{c^2 a^3} \left[ \frac{3}{a^2} a^2 naJ \sin i \sin \nu - anJ \sin i \sin \nu \right] = 4 \frac{GM}{c^2 a^2} nJ \sin i \sin \nu \end{aligned} \quad (19)$$

The constant radial component causes a reduction of the semimajor axis, according to (14), of

$$\Delta a = -\frac{2}{3} \frac{anJ}{c^2} \cos i \propto a^{-1/2}. \quad (20)$$

The effect on the semimajor axis decreases with the square root of the semimajor axis. Note the factor  $\cos i$  in the radial acceleration and in the change of the semimajor axis. As a consequence there is no radial acceleration and no change of semimajor axis for orbits perpendicular to the equatorial plane.

According to (5) we obtain a perturbation in the right ascension of the ascending node (using  $u = \nu$  for circular orbit)

$$\dot{\Omega}_{\text{LT}} = \frac{\sin u}{na \sin i} \cdot W = 4 \frac{GM}{c^2 a^3} J \sin^2 u = 2 \frac{GM}{c^2 a^3} J (1 - \cos 2u) \quad (21)$$

that includes a secular term. The orbit precession is independent on the orbit inclination  $i$ , as expected, since the effect is a frame precession. The precession is proportional to  $a^{-3}$ , i.e., drops rapidly with increasing orbit radius.

### 3 DeSitter

The deSitter or geodetic precession is caused by the rotation of the Earth around the Sun. Space curvature causes a precession of a parallelly transported (local inertial) frame with respect to distant stars. In the vicinity of the Earth the rate of precession amounts to 19.2 mas/y and does not depend on the distance from the Earth. The corresponding acceleration contains the mass  $M_S$  and distance  $R$  of the Sun and has the form of a Coriolis term (see (1)):

$$\Delta \ddot{\vec{r}}_{\text{dS}} = -2 \left[ -\frac{3}{2} \frac{GM_S}{c^2 R^3} \vec{R} \times \dot{\vec{R}} \right] \times \vec{v}. \quad (22)$$

The term in brackets is the precession vector  $\vec{\omega}_{\text{dS}}$  of the global frame (used to represent the satellite orbit) with respect to the local inertial (precessing) frame. It points to the southern ecliptic pole and its magnitude is  $2.95 \cdot 10^{-15}$  rad/s or 53  $\mu\text{as/d}$  resp. 19.2 mas/yr.

Redefining the frame used in (8) (x-axis towards ascending node with respect to ecliptic plane, same z-axis) we may write

$$\vec{\omega}_{\text{dS}} = -\frac{3}{2} \frac{GM_S}{c^2 R^3} \vec{R} \times \dot{\vec{R}} = -\frac{3}{2} \frac{GM_S}{c^2 R} \sqrt{1 - e_S^2} n_S (0, \sin \beta, \cos \beta). \quad (23)$$

$n_S$  is the mean motion of the Earth around the Sun. For the eccentricity  $e_S=0.01673$  of the Earth's orbit the square root term is  $\sqrt{1 - e_S^2} = 1 - 1.40 \cdot 10^{-4}$ .  $\beta$  is the inclination of the orbital plane with respect to the ecliptic. It is in the range  $i - \varepsilon$  and  $i + \varepsilon$ , where  $\varepsilon=23.5$  deg is the obliquity of the ecliptic and may be computed with

$$\cos \beta = \cos \varepsilon \cos i + \sin \varepsilon \sin i \cos \Omega. \quad (24)$$

Interestingly the associated centrifugal term  $-\vec{\omega}_{\text{dS}} \times (\vec{\omega}_{\text{dS}} \times \vec{r})$  is missing in (22). The reason is its magnitude of at maximum (for a geosynchronous orbit)

$$\left( \frac{3}{2} \frac{GM_S}{c^2 R} n_S \right)^2 a \simeq 4 \cdot 10^{-22} \text{ m/s}^2. \quad (25)$$

In the defined frame we may write for a circular orbit

$$(\vec{R} \times \dot{\vec{R}}) \times \vec{v} = R^2 n_S \sqrt{1 - e_S^2} a (-\cos \beta \cos \nu, -\cos \beta \sin \nu, \sin \beta \sin \nu). \quad (26)$$

With this we may finally write the perturbing accelerations in radial, alongtrack and crosstrack directions

$$\begin{aligned} R &= \Delta \ddot{\vec{r}}_{\text{dS}} \cdot \vec{e}_R = -3 \frac{GM_S}{c^2 R} n_S \sqrt{1 - e_S^2} a \cos \beta \\ S &= \Delta \ddot{\vec{r}}_{\text{dS}} \cdot \vec{e}_S = 0 \\ W &= \Delta \ddot{\vec{r}}_{\text{dS}} \cdot \vec{e}_W = 3 \frac{GM_S}{c^2 R} n_S \sqrt{1 - e_S^2} a \sin \beta \sin \nu \end{aligned} \quad (27)$$

Note again the factor  $\cos \beta$  in the radial acceleration. For orbits perpendicular to the ecliptic plane there is no radial acceleration (and hence no change in semimajor axis) because the Coriolis acceleration is perpendicular to the orbital plane.

The constant negative radial acceleration causes an increase of the semimajor axis by

$$\Delta a = + \frac{GM_S}{c^2} \frac{a}{R} \frac{n_S}{n} \sqrt{1 - e_S^2} \cos \beta \propto a^{5/2}. \quad (28)$$

$\Delta a$  is increasing with in creasing distance from the Earth. With (5) we compute the precession of the ascending node with respect to the ecliptic plane (again using  $u = \nu$  for circular orbit)

$$\dot{\Omega}_{\text{dS}} = \frac{\sin u}{na \sin \beta} \cdot W = 3 \frac{GM_S}{c^2 R} n_S \sqrt{1 - e_S^2} \sin^2 u = \frac{3}{2} \frac{GM_S}{c^2 R} n_S \sqrt{1 - e_S^2} (1 - \cos 2u). \quad (29)$$

It is no surprise that the secular term is equal to the precession rate given in (23).

## 4 Orders of Magnitude

The following Table 1 gives orders of magnitude for different terms and different satellites. The Lense-Thirring acceleration is for all orbit types smaller than the Schwarzschild acceleration by two orders of magnitude. For semimajor axes lower than about 11000 km the Lense-Thirring effect is more important than the deSitter effect. It is probably no coincidence that the accelerations and orbit precession rates due to Lense-Thirring and deSitter are of equal magnitude for LAGEOS.

For semimajor axes above 11000 km the deSitter effect is more important than Lense-Thirring. For geosynchronous satellites the deSitter acceleration is only one order of magnitude smaller than the Schwarzschild acceleration. At a distance larger than about 73000 km it even becomes more important than the Schwarzschild effect, probably marking the limit of validity of the underlying assumptions.

Term	GEO	GPS	LAGEOS	Jason	CHAMP	
Height	35786	20184	5850	1335	350	km
Schwarzschild						
$3\frac{(GM)^2}{c^2 a^3}$	$7.07 \cdot 10^{-11}$	$2.83 \cdot 10^{-10}$	$2.90 \cdot 10^{-9}$	$1.16 \cdot 10^{-8}$	$1.74 \cdot 10^{-8}$	$\text{ms}^{-2}$
$\Delta a$	-4.4	-4.4	-4.4	-4.4	-4.4	mm
Lense-Thirring						
$2\frac{GM}{c^2 a^2} n J$	$3.57 \cdot 10^{-13}$	$1.80 \cdot 10^{-12}$	$2.71 \cdot 10^{-11}$	$1.36 \cdot 10^{-10}$	$2.20 \cdot 10^{-10}$	$\text{ms}^{-2}$
$\Delta a / \cos i$	-22	-28	-42	-52	-56	$\mu\text{m}$
$\dot{\Omega}$	2.1	8.8	85	340	510	$\mu\text{as/d}$
deSitter						
$3\frac{GM_s}{c^2 R} n_{San}$	$1.81 \cdot 10^{-11}$	$2.28 \cdot 10^{-11}$	$3.37 \cdot 10^{-11}$	$4.24 \cdot 10^{-11}$	$4.54 \cdot 10^{-11}$	$\text{m/s}^2$
$\Delta a / \cos \beta$	1100	360	51	16	12	$\mu\text{m}$
$\dot{\Omega}$	53	53	53	53	53	$\mu\text{as/d}$

Table 1: Orders of magnitude for different terms and different satellites

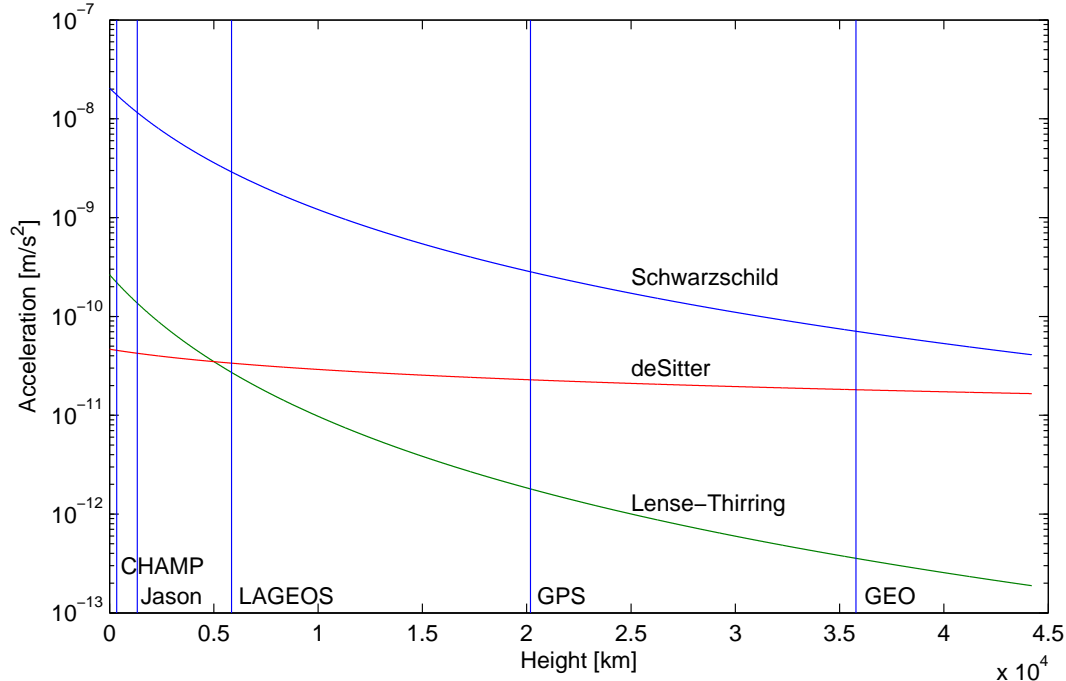


Figure 1: Radial acceleration as function of height for circular orbits with vanishing inclination